Eur. Phys. J. C 23, 121–133 (2002)
Digital Object Identifier (DOI) 10.1007/s100520100825

THE EUROPEAN
PHYSICAL JOURNAL C

Radiative corrections to $K_{\ell 3}$ decays*

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Received: 12 October 2001 /
Published online: 11 January 2002 –

Abstract. We present a complete calculation of the $K_{\ell 3}$ decays $K^+ \to \pi^0 \ell^+ \nu_\ell$ and $K^0 \to \pi^- \ell^+ \nu_\ell$ to $\mathcal{O}(p^4, (m_d - m_u)p^2, e^2 p^2)$ in chiral perturbation theory with virtual photons and leptons. We introduce the concept of generalized form factors and kinematical densities in the presence of electromagnetism, and propose a possible treatment of the real photon emission in $K^+_{\ell 3}$ decays. We illustrate our results by applying them to the extraction of the Kobayashi–Maskawa matrix element $|V_{us}|$ from the experimental K^+_{e3} decay parameters.

1 Introduction

Semileptonic kaon decays have played a central role in our understanding of flavour physics. In particular, the K_{e3} decay mode is often presented as the most accurate and theoretically cleanest source for the extraction of the Kobayashi–Maskawa matrix element $|V_{us}|$ [1]. However, as new high precision experiments are planned and the theoretical tool of effective field theory has been pushed to higher orders, an update of the theoretical analysis of such decays is needed. In particular, it is interesting to improve as much as possible the theoretical analysis underlying the extraction of $|V_{us}|$. This is presently based on the following ingredients: a calculation of the hadronic form factors [2] at order p^4 in chiral perturbation theory (CHPT) [3,4], supplemented by a model-dependent estimate of the order p^6 effects [5]. As for the radiative corrections, the model independent short distance leading logarithms [6,7] have been included, together with a model dependent estimate of the long distance contributions [8–11].

It is the purpose of this paper to work out the radiative corrections to the $K_{\ell 3}$ decays ($\ell = e, \mu$) within the framework of CHPT, the effective theory of the standard model at low energy. The appropriate formalism for including virtual photons in purely mesonic low-energy processes has been presented in [12–14]. This scheme was then extended for the treatment of semileptonic interactions by the additional inclusion of virtual leptons [15]. The goal of our analysis is to obtain the size of such corrections and quantify the theoretical uncertainty to be assigned to them. This will allow us to understand how well one can know $|V_{us}|$ once new high statistics experiments collect data and the full order p^6 CHPT analysis becomes available [16].

The outline of this paper is as follows. In Sect. 2 we recapitulate CHPT in the presence of virtual photons and leptons. The basic $K_{\ell 3}$ phenomenology in the absence of electromagnetism is reviewed in Sect. 3. The structure of the radiative corrections to $K_{\ell 3}$ decays is discussed in Sect. 4. The $K_{\ell 3}$ amplitudes to $\mathcal{O}(p^4)$ and $\mathcal{O}(e^2p^2)$ are calculated in Sect. 5. In Sect. 6 we propose a specific treatment of the real photon emission in the $K^+_{\ell 3}$ case. In Sect. 7 we illustrate our general considerations by a numerical study of the K^+_{e3} decay and the description of a procedure to extract the Kobayashi–Maskawa matrix element $|V_{us}|$ from experimental data. Our conclusions are summarized in Sect. 8. Appendix A contains a summary of integrals appearing in various mesonic one-loop amplitudes and Appendix B collects several photonic loop functions. In Appendix C we report a set of coefficients appearing in the next-to-leading order expansion of the form factors.

2 The standard model at low energies

The appropriate theoretical framework for the analysis of electromagnetic effects in semileptonic kaon decays is a low-energy effective quantum field theory where the asymptotic states consist of the pseudoscalar octet, the photon and the light leptons [15]. The corresponding lowest order effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\langle u_\mu u^\mu + \chi_+ \rangle + e^2 F_0^4 Z \langle \mathcal{Q}^{\text{em}}_{\text{L}} \mathcal{Q}^{\text{em}}_{\text{R}} \rangle - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_\ell [\overline{\ell}(\mathrm{i}\,\partial\!\!\!/ + e A\!\!\!/ - m_\ell)\ell + \overline{\nu_{\ell \text{L}}}\mathrm{i}\,\partial\!\!\!/ \nu_{\ell \text{L}}]. \tag{2.1}$$

* Work supported in part by TMR, EC-Contract No. ERBFMRX-CT980169 (EURODAΦNE)

The symbol $\langle\ \rangle$ denotes the trace in three-dimensional flavour space, and

$$u_\mu = \mathrm{i}[u^\dagger(\partial_\mu - \mathrm{i}r_\mu)u - u(\partial_\mu - \mathrm{i}l_\mu)u^\dagger]. \tag{2.2}$$

The photon field A_μ and the leptons ℓ, ν_ℓ ($\ell = e, \mu$) are contained in (2.2) by adding appropriate terms to the usual external vector and axial-vector sources v_μ, a_μ:

$$\begin{aligned} l_\mu &= v_\mu - a_\mu - eQ_{\mathrm{L}}^{\mathrm{em}} A_\mu + \sum_\ell (\bar\ell \gamma_\mu \nu_{\ell\mathrm{L}} Q_{\mathrm{L}}^{\mathrm{w}} + \overline{\nu_{\ell\mathrm{L}}} \gamma_\mu \ell Q_{\mathrm{L}}^{\mathrm{w}\dagger}), \\ r_\mu &= v_\mu + a_\mu - eQ_{\mathrm{R}}^{\mathrm{em}} A_\mu. \end{aligned} \tag{2.3}$$

The 3×3 matrices $Q_{\mathrm{L,R}}^{\mathrm{em}}$, $Q_{\mathrm{L}}^{\mathrm{w}}$ are spurion fields. At the end, one identifies $Q_{\mathrm{L,R}}^{\mathrm{em}}$ with the quark charge matrix

$$Q^{\mathrm{em}} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}, \tag{2.4}$$

whereas the weak spurion is taken at

$$Q_{\mathrm{L}}^{\mathrm{w}} = -2\sqrt{2} G_{\mathrm{F}} \begin{bmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.5}$$

where G_{F} is the Fermi coupling constant and V_{ud}, V_{us} are Kobayashi–Maskawa matrix elements. For the construction of the effective Lagrangian it is also convenient to define

$$\mathcal{Q}_{\mathrm{L}}^{\mathrm{em,w}} := u Q_{\mathrm{L}}^{\mathrm{em,w}} u^\dagger, \quad \mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} := u^\dagger Q_{\mathrm{R}}^{\mathrm{em}} u. \tag{2.6}$$

F_0 denotes the pion decay constant in the chiral limit and in the absence of electroweak interactions. Explicit chiral symmetry breaking is included in $\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u$ where χ is proportional to the quark mass matrix:

$$\chi = 2B_0 \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}. \tag{2.7}$$

The factor B_0 is related to the quark condensate in the chiral limit by $\langle 0|\overline{q}q|0\rangle = -F_0^2 B_0$. This leads to the (lowest order) expressions for the pseudoscalar masses

$$\begin{aligned} M_{\pi^\pm}^2 &= 2B_0\widehat{m} + 2e^2 Z F_0^2, \\ M_{\pi^0}^2 &= 2B_0\widehat{m}, \\ M_{K^\pm}^2 &= B_0\left[(m_s + \widehat{m}) - \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s - \widehat{m})\right] + 2e^2 Z F_0^2, \\ M_{\overset{(-)}{K}{}^0}^2 &= B_0\left[(m_s + \widehat{m}) + \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s - \widehat{m})\right], \\ M_\eta^2 &= \frac{4}{3}B_0\left(m_s + \frac{\widehat{m}}{2}\right). \end{aligned} \tag{2.8}$$

The tree-level mixing angle $\varepsilon^{(2)}$ is given by

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\frac{m_d - m_u}{m_s - \widehat{m}}, \tag{2.9}$$

the symbol $\widehat{m}$ stands for the mean value of the light quark masses,

$$\widehat{m} = \frac{1}{2}(m_u + m_d). \tag{2.10}$$

For later use, we also denote the isospin limits ($m_u = m_d$, $e = 0$) of the pion mass and the kaon mass, respectively, by

$$M_\pi^2 = 2B_0\widehat{m}, \quad M_K^2 = B_0(m_s + \widehat{m}). \tag{2.11}$$

Using (2.8), the numerical value of the coupling constant $Z \simeq 0.8$ can be determined from the mass difference of the charged pions.

In our calculation we include terms in the low-energy expansion up to order $p^4, (m_u - m_d)p^2$ and e^2p^2.

The most general local action at next-to-leading order can be written as the sum of four terms, $\mathcal{L}_{p^4} + \mathcal{L}_{e^2p^2} + \mathcal{L}_{\mathrm{lept}} + \mathcal{L}_\gamma$. The first one, $\mathcal{L}_{p^4}$ is the well-known Gasser–Leutwyler Lagrangian [4] in the presence of the generalized external sources introduced in (2.3):

$$\begin{aligned} \mathcal{L}_{p^4} &= L_1\langle u_\mu u^\mu\rangle^2 + L_2\langle u_\mu u^\nu\rangle\langle u^\mu u_\nu\rangle \\ &+ L_3\langle u_\mu u^\mu u_\nu u^\nu\rangle + L_4\langle u_\mu u^\mu\rangle\langle\chi_+\rangle \\ &+ L_5\langle u_\mu u^\mu \chi_+\rangle + L_6\langle\chi_+\rangle^2 + L_7\langle\chi_-\rangle^2 \\ &+ \frac{1}{4}(2L_8 + L_{12})\langle\chi_+^2\rangle + \frac{1}{4}(2L_8 - L_{12})\langle\chi_-^2\rangle \\ &- \mathrm{i}L_9\langle f_+^{\mu\nu} u_\mu u_\nu\rangle + \frac{1}{4}(L_{10} + 2L_{11})\langle f_{+\mu\nu} f_+^{\mu\nu}\rangle \\ &- \frac{1}{4}(L_{10} - 2L_{11})\langle f_{-\mu\nu} f_-^{\mu\nu}\rangle, \end{aligned} \tag{2.12}$$

with

$$\begin{aligned} f_\pm^{\mu\nu} &= u F_{\mathrm{L}}^{\mu\nu} u^\dagger \pm u^\dagger F_{\mathrm{R}}^{\mu\nu} u, \\ F_{\mathrm{L}}^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - \mathrm{i}[l^\mu, l^\nu], \\ F_{\mathrm{R}}^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - \mathrm{i}[r^\mu, r^\nu]. \end{aligned} \tag{2.13}$$

The second term, $\mathcal{L}_{e^2p^2}$, denotes the interaction of dynamical photons with hadronic degrees of freedom [12–14]. Its expression is

$$\begin{aligned} \mathcal{L}_{e^2p^2} = e^2F_0^2 \Big\{ &\frac{1}{2}K_1\langle(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})^2 + (\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}})^2\rangle\langle u_\mu u^\mu\rangle \\ &+ K_2\langle\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\rangle\langle u_\mu u^\mu\rangle \\ &- K_3[\langle\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}} u_\mu\rangle\langle\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}} u^\mu\rangle + \langle\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} u_\mu\rangle\langle\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} u^\mu\rangle] \\ &+ K_4\langle\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}} u_\mu\rangle\langle\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} u^\mu\rangle \\ &+ K_5\langle[(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})^2 + (\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}})^2] u_\mu u^\mu\rangle \\ &+ K_6\langle(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} + \mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}) u_\mu u^\mu\rangle \\ &+ \frac{1}{2}K_7\langle(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})^2 + (\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}})^2\rangle\langle\chi_+\rangle \\ &+ K_8\langle\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\rangle\langle\chi_+\rangle \\ &+ K_9\langle[(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})^2 + (\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}})^2]\chi_+\rangle \\ &+ K_{10}\langle(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} + \mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})\chi_+\rangle \\ &- K_{11}\langle(\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} - \mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})\chi_-\rangle \\ &- \mathrm{i}K_{12}\langle[(\widehat{\nabla}_\mu\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}})\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}} - \mathcal{Q}_{\mathrm{L}}^{\mathrm{em}}\widehat{\nabla}_\mu\mathcal{Q}_{\mathrm{L}}^{\mathrm{em}} \\ &\quad - (\widehat{\nabla}_\mu\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}})\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}} + \mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}\widehat{\nabla}_\mu\mathcal{Q}_{\mathrm{R}}^{\mathrm{em}}] u^\mu\rangle \end{aligned}$$

$$
\left.\begin{aligned}
&+ K_{13}\langle(\widehat{\nabla}_\mu \mathcal{Q}_{\rm L}^{\rm em})(\widehat{\nabla}^\mu \mathcal{Q}_{\rm R}^{\rm em})\rangle \\
&+ K_{14}\langle(\widehat{\nabla}_\mu \mathcal{Q}_{\rm L}^{\rm em})(\widehat{\nabla}^\mu \mathcal{Q}_{\rm L}^{\rm em}) \\
&\quad + (\widehat{\nabla}_\mu \mathcal{Q}_{\rm R}^{\rm em})(\widehat{\nabla}^\mu \mathcal{Q}_{\rm R}^{\rm em})\rangle
\end{aligned}\right\}, \tag{2.14}
$$

where

$$
\begin{aligned}
\widehat{\nabla}_\mu \mathcal{Q}_{\rm L}^{\rm em} &= \nabla_\mu \mathcal{Q}_{\rm L}^{\rm em} + \frac{\rm i}{2}[u_\mu, \mathcal{Q}_{\rm L}^{\rm em}] \\
&= u D_\mu Q_{\rm L}^{\rm em} u^\dagger, \\
\widehat{\nabla}_\mu \mathcal{Q}_{\rm R}^{\rm em} &= \nabla_\mu \mathcal{Q}_{\rm R}^{\rm em} - \frac{\rm i}{2}[u_\mu, \mathcal{Q}_{\rm R}^{\rm em}] \\
&= u^\dagger D_\mu Q_{\rm R}^{\rm em} u,
\end{aligned} \tag{2.15}
$$

with

$$
\begin{aligned}
D_\mu Q_{\rm L}^{\rm em} &= \partial_\mu Q_{\rm L}^{\rm em} - {\rm i}[l_\mu, Q_{\rm L}^{\rm em}], \\
D_\mu Q_{\rm R}^{\rm em} &= \partial_\mu Q_{\rm R}^{\rm em} - {\rm i}[r_\mu, Q_{\rm R}^{\rm em}].
\end{aligned} \tag{2.16}
$$

The presence of virtual leptons requires also the addition of the "leptonic" term [15]

$$
\begin{aligned}
\mathcal{L}_{\rm lept} = e^2 \sum_\ell \Big\{ & {F_0}^2 \Big[X_1 \bar{\ell}\gamma_\mu \nu_{\ell L} \langle u^\mu \{\mathcal{Q}_{\rm R}^{\rm em}, \mathcal{Q}_{\rm L}^{\rm w}\}\rangle \\
&+ X_2 \bar{\ell}\gamma_\mu \nu_{\ell L}\langle u^\mu[\mathcal{Q}_{\rm R}^{\rm em}, \mathcal{Q}_{\rm L}^{\rm w}]\rangle + X_3 m_\ell \bar{\ell}\nu_{\ell L}\langle \mathcal{Q}_{\rm L}^{\rm w}\mathcal{Q}_{\rm R}^{\rm em}\rangle \\
&+ {\rm i}X_4 \bar{\ell}\gamma_\mu \nu_{\ell L}\langle \mathcal{Q}_{\rm L}^{\rm w}\widehat{\nabla}^\mu \mathcal{Q}_{\rm L}^{\rm em}\rangle + {\rm i}X_5 \bar{\ell}\gamma_\mu \nu_{\ell L}\langle \mathcal{Q}_{\rm L}^{\rm w}\widehat{\nabla}^\mu \mathcal{Q}_{\rm R}^{\rm em}\rangle \\
&+ {\rm h.c.}\Big] + X_6 \bar{\ell}({\rm i}\,\partial\!\!\!/ + e A\!\!\!/)\ell + X_7 m_\ell \bar{\ell}\ell \Big\}.
\end{aligned} \tag{2.17}
$$

In $\mathcal{L}_{\rm lept}$ we consider only terms quadratic in the lepton fields and at most linear in $G_{\rm F}$. Finally, also a photon Lagrangian

$$
\mathcal{L}_\gamma = e^2 X_8 F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.18}
$$

has to be added [15].

The low-energy couplings L_i, K_i, X_i arising here are divergent (except L_3, L_7, K_7, K_{13}, K_{14} and X_1). They absorb the poles of the one-loop graphs via the renormalization

$$
\begin{aligned}
L_i &= L_i^r(\mu) + \Gamma_i \Lambda(\mu), \quad i = 1, \ldots, 12, \\
K_i &= K_i^r(\mu) + \Sigma_i \Lambda(\mu), \quad i = 1, \ldots, 14, \\
X_i &= X_i^r(\mu) + \Xi_i \Lambda(\mu), \quad i = 1, \ldots, 8,
\end{aligned} \tag{2.19}
$$

with

$$
\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2}\left\{\frac{1}{d-4} - \frac{1}{2}[\ln(4\pi) + \Gamma'(1) + 1]\right\} \tag{2.20}
$$

in the dimensional regularization scheme. The coefficients Γ_i and Σ_i can be found in [4] and in [12], respectively. Their values are not modified by the presence of virtual leptons as long as contributions of $\mathcal{O}(G_{\rm F}^2)$ are neglected. The "leptonic" coefficients Ξ_i have been determined rather recently in [15] by using super-heat-kernel techniques [17].

In order to match our low-energy effective theory to the standard model of strong and electroweak interactions, we have to specify the precise physical meaning of the parameter $G_{\rm F}$. In the presence of electromagnetism, this identification is somewhat ambiguous. The shift

$$
G_{\rm F} \to G_{\rm F}(1 + e^2\delta) \tag{2.21}
$$

induces the change (see (4.8) of [15])

$$
\frac{F_0^2}{4}\langle u_\mu u^\mu\rangle \to \frac{F_0^2}{4}\langle u_\mu u^\mu\rangle - 2e^2\delta \sum_\ell \bar{\ell}({\rm i}\,\partial\!\!\!/ + eA\!\!\!/ - m_\ell)\ell, \tag{2.22}
$$

corresponding to

$$
X_6 \to X_6 - 2\delta, \quad X_7 \to X_7 + 2\delta. \tag{2.23}
$$

In other words, some part of the electromagnetic contributions may always be shuffled from X_6 to $G_{\rm F}$ or vice versa. (The coupling constant X_7 does not appear in observable quantities as it is always absorbed by the mass renormalization of the charged leptons.)

Following [15], we identify $G_{\rm F}$ with the muon decay constant. To order α, $G_{\rm F}$ can be related to the measured muon decay width by [18]

$$
\begin{aligned}
\Gamma(\mu \to {\rm all}) &= \frac{G_{\rm F}^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right) \\
&\quad \times \left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right) + \mathcal{O}(\alpha^2)\right],
\end{aligned} \tag{2.24}
$$

with $f(x) = 1 - 8x - 12x^2 \ln x + 8x^3 - x^4$. With this choice of $G_{\rm F}$, the (universal) short distance electromagnetic correction [6,7] of semileptonic charged current amplitudes is fully contained in the coupling constant X_6. To display this dependence explicitly, we pull out the short distance part $X_6^{\rm SD}$ by the decomposition

$$
X_6^r(\mu) = X_6^{\rm SD} + \widetilde{X}_6^r(\mu), \tag{2.25}
$$

where

$$
e^2 X_6^{\rm SD} = -\frac{e^2}{4\pi^2}\log\frac{M_Z^2}{M_\rho^2} = 1 - S_{\rm EW}(M_\rho, M_Z), \tag{2.26}
$$

which defines [7] also the short distance enhancement factor $S_{\rm EW}(M_\rho, M_Z)$ (to leading order). With this splitting of X_6, we expect its "long distance part" $\widetilde{X}_6^r(M_\rho)$ to have the typical size of a low-energy constant in CHPT.

3 Basic $K_{\ell 3}$ phenomenology

In this section we report the basic formulae of $K_{\ell 3}$ phenomenology in the absence of radiative corrections[1].

[1] In the present section M_K and M_π stand for the physical masses of the corresponding mesons involved in the process

3.1 Invariant amplitude

We shall consider the generic $K_{\ell 3}$ process

$$K(p_K) \to \pi(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu). \quad (3.1)$$

The invariant amplitude reads

$$\mathcal{M} = \frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us}^* l^\mu C \times \left[f_+^{K\pi}(t)(p_K + p_\pi)_\mu + f_-^{K\pi}(t)(p_K - p_\pi)_\mu \right], \quad (3.2)$$

where

$$l^\mu = \bar{u}(p_\nu)\gamma^\mu(1-\gamma_5)v(p_\ell), \quad C = \begin{cases} 1 & \text{for } K^0_{\ell 3}, \\ \frac{1}{\sqrt{2}} & \text{for } K^+_{\ell 3}. \end{cases}$$

The expression in parentheses corresponds to the matrix element $\langle \pi(p_\pi)|V_\mu^{4-\mathrm{i}5}|K(p_K)\rangle$, expressed in terms of the form factors $f_\pm^{K\pi}(t)$. The hadronic form factors depend on the single variable $t = (p_K - p_\pi)^2$.

3.2 Dalitz plot density

It is customary to analyze the spin-averaged decay distribution $\rho(y,z)$ for $K_{\ell 3}$. It depends on two variables, for which we choose

$$z = \frac{2p_\pi \cdot p_K}{M_K^2} = \frac{2E_\pi}{M_K}, \quad y = \frac{2p_K \cdot p_\ell}{M_K^2} = \frac{2E_\ell}{M_K}, \quad (3.3)$$

where E_π (E_ℓ) is the pion (charged lepton) energy in the kaon rest frame, and M_K indicates the mass of the decaying kaon. Alternatively one may also use two of the Lorentz invariants

$$t = (p_K - p_\pi)^2, \quad u = (p_K - p_\ell)^2, \quad s = (p_\pi + p_\ell)^2. \quad (3.4)$$

Then the distribution (without radiative corrections) reads

$$\rho^{(0)}(y,z) = \mathcal{N}\Big[A_1^{(0)}|f_+^{K\pi}(t)|^2 + A_2^{(0)} f_+^{K\pi}(t) f_-^{K\pi}(t) + A_3^{(0)}|f_-^{K\pi}(t)|^2 \Big], \quad (3.5)$$

with

$$\mathcal{N} = C^2 \frac{G_{\mathrm{F}}^2 |V_{us}|^2 M_K^5}{128\pi^3}, \quad \Gamma = \int_{\mathcal{D}} \mathrm{d}y\mathrm{d}z \rho^{(0)}(y,z). \quad (3.6)$$

Moreover, defining

$$r_\ell = \frac{m_\ell^2}{M_K^2}, \quad r_\pi = \frac{M_\pi^2}{M_K^2}, \quad (3.7)$$

the kinematical densities are

$$\begin{aligned} A_1^{(0)}(y,z) &= 4(z+y-1)(1-y) + r_\ell(4y+3z-3) \\ &\quad - 4r_\pi + r_\ell(r_\pi - r_\ell), \\ A_2^{(0)}(y,z) &= 2r_\ell(3-2y-z+r_\ell-r_\pi), \\ A_3^{(0)}(y,z) &= r_\ell(1+r_\pi-z-r_\ell). \end{aligned} \quad (3.8)$$

In the analysis of K_{e3} decays, the terms proportional to $A_{2,3}^{(0)}$ can be neglected, being proportional to $r_e \simeq 10^{-6}$. The physical domain $\mathcal{D}$ is defined by

$$\begin{aligned} 2\sqrt{r_\ell} \le y \le 1 + r_\ell - r_\pi, \\ a(y) - b(y) \le z \le a(y) + b(y), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} a(y) &= \frac{(2-y)(1+r_\ell+r_\pi-y)}{2(1+r_\ell-y)}, \\ b(y) &= \frac{\sqrt{y^2-4r_\ell}(1+r_\ell-r_\pi-y)}{2(1+r_\ell-y)}, \end{aligned} \quad (3.10)$$

or, equivalently,

$$\begin{aligned} 2\sqrt{r_\pi} \le z \le 1 + r_\pi - r_\ell, \\ c(z) - d(z) \le y \le c(z) + d(z), \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} c(z) &= \frac{(2-z)(1+r_\pi+r_\ell-z)}{2(1+r_\pi-z)}, \\ d(z) &= \frac{\sqrt{z^2-4r_\pi}(1+r_\pi-r_\ell-z)}{2(1+r_\pi-z)}. \end{aligned} \quad (3.12)$$

4 Structure of radiative corrections to $K_{\ell 3}$ decays

In decays involving charged particles, the observable quantity always involves an inclusive sum over the *parent* mode and final states with additional photons. Therefore, when considering radiative corrections one must include the effect of real photon radiation as well as virtual electromagnetic corrections. Moreover, from a theoretical point of view, only such an inclusive sum is free of infrared (IR) divergences order by order in α. In the present section we give an overview on the analysis of $K_{\ell 3}$ decays with inclusion of radiative corrections. We show how electromagnetic corrections can be accounted for by using generalized form factors and kinematical densities, whose specific form we describe in detail in the following sections.

The virtual corrections will in general produce a shift in the invariant amplitude $\mathcal{M} \to \mathcal{M} + \Delta\mathcal{M}$. On the other hand, the radiation of real photons (governed by an amplitude $\mathcal{M}^\gamma$) will produce a shift to the decay distribution and rate.

The virtual electromagnetic corrections do not alter the structure of the invariant amplitude (3.2) in terms of the form factors $f_\pm(t)$, but change the form factors themselves. We denote by $F_\pm(t,v)$ the full form factors including virtual electromagnetic corrections. Note that such objects now depend also on a second kinematical variable (this is because $F_\pm$ cannot be interpreted anymore as matrix elements of a quark current between hadronic states). The variable v is taken as $u = (p_K - p_\ell)^2$ for $K^+_{\ell 3}$ and $s = (p_\pi + p_\ell)^2$ for $K^0_{\ell 3}$. We observe here that $F_\pm$ contain infrared singularities, due to low-momentum virtual photons: we regularize them by introducing a small photon mass M_γ. Summarizing, the effect of virtual corrections is to change the invariant amplitude (3.2) as follows:

$$\mathcal{M}\big[f_+^{(0)}(t), f_-^{(0)}(t)\big] \to \mathcal{M}\big[F_+(t,v), F_-(t,v)\big]. \quad (4.1)$$

Moreover, it is convenient to factor out of $F_\pm$ the long distance component $\Gamma_c(v, m_\ell^2, M^2; M_\gamma)$ of loop amplitudes, which generates infrared and Coulomb singularities, as follows:

$$F_\pm(t,v) = \left[1 + \frac{\alpha}{4\pi}\Gamma_c(v, m_\ell^2, M^2; M_\gamma)\right] f_\pm(t,v). \quad (4.2)$$

Γ_c depends on the mass m_ℓ of the charged lepton, the mass M of the charged meson, the variable v and has a logarithmic dependence with respect to the infrared regulator M_γ. The definition of Γ_c is not unique (due to possible shifts in the infrared finite parts), and we give our definition in (5.1). Once one specifies the form of Γ_c, (4.2) serves also as a definition of the structure-dependent effective form factors $f_\pm(t,v)$. In the next section we shall present the calculation of $f_\pm$ at next-to-leading order in CHPT.

Let us now comment on the radiative amplitude $\mathcal{M}^\gamma$, and how it affects the decay distribution. In the intermediate stages of this work we shall use Low's theorem [19] to obtain the leading components of $\mathcal{M}^\gamma$ at small photon momentum. This is consistent with a full calculation at order e^2p^2 in CHPT[2]. However, it allows us to be more general in the derivation of the radiatively corrected decay distribution, by including at intermediate steps some higher order terms in the chiral counting. We restore the correct chiral counting by inserting in the final expression for the decay rate the form factors as calculated at $\mathcal{O}(p^4, e^2p^2)$ in CHPT. Let us now sketch the analysis. The radiative amplitude can be expanded in powers of the photon energy E_γ,

$$\mathcal{M}^\gamma = \mathcal{M}^\gamma_{(-1)} + \mathcal{M}^\gamma_{(0)} + \cdots, \quad (4.3)$$

where

$$\mathcal{M}^\gamma_{(n)} \sim E_\gamma^n. \quad (4.4)$$

Gauge invariance relates $\mathcal{M}^\gamma_{(-1)}$ and $\mathcal{M}^\gamma_{(0)}$ to the non-radiative amplitude $\mathcal{M}$, and thus to the full form factors $F_\pm(t,v)$. Upon taking the square modulus and summing over spins, the radiative amplitude generates a correction $\rho_\gamma(y,z)$ to the Dalitz plot density of (3.5). The observable distribution is now the sum

$$\rho(y,z) = \rho^{(0)}(y,z) + \rho_\gamma(y,z). \quad (4.5)$$

Both terms on the right hand side of this equation depend on the full form factors $F_\pm$ and contain infrared divergences (from virtual or real soft photons). Upon combining them, the observable density can be written in terms of new kinematical densities A_i and the effective form factors $f_\pm(t,v)$ defined in (4.2),

$$\rho(y,z) = \mathcal{N} S_{\rm EW}(M_\rho, M_Z) \quad (4.6)$$
$$\times[A_1|f_+(t,v)|^2 + A_2 f_+(t,v) f_-(t,v) + A_3|f_-(t,v)|^2],$$

[2] At this order CHPT reproduces the results of Low's theorem, with $f_+(t) = 1$ and $f_-(t) = 0$

where we have pulled out the short distance enhancement factor $S_{\rm EW}$ discussed in Sect. 2. To first order in α, the kinematical densities are given by

$$A_i(y,z) = A_i^{(0)}(y,z)\left[1 + \Delta^{\rm IR}(y,z)\right] + \Delta_i^{\rm IB}(y,z). \quad (4.7)$$

The function $\Delta^{\rm IR}(y,z)$ arises by combining the contributions from $|\mathcal{M}^\gamma_{(-1)}|^2$ and $\Gamma_c(v, m_\ell^2, M^2; M_\gamma)$. Although the individual contributions contain infrared divergences, the sum is finite. The factors $\Delta_i^{\rm IB}(y,z)$ originate from averaging the remaining terms of $|\mathcal{M}^\gamma|^2$ [see (4.3)] and are IR finite. Note that both $\Delta^{\rm IR}(y,z)$ and $\Delta_i^{\rm IB}(y,z)$ are sensitive to the treatment of the real photon emission. Details on these corrections are given in Sect. 6.
Let us finally note that, in principle, the radiative amplitude generates new terms in the density, proportional to derivatives of form factors. These terms would only arise at order e^2p^4 and higher in CHPT, and therefore we have suppressed them in (4.6).

Summarizing, the resulting Dalitz density is formally identical to the unperturbed one, but now one has

$$f_\pm^{(0)}(t) \to f_\pm(t,v),$$
$$A_i^{(0)}(y,z) \to A_i(y,z). \quad (4.8)$$

We want to stress here that although the structure dependent electromagnetic effects in $f_\pm(t,v)$ are due to the interplay between QCD dynamics and QED, the corrections of order α included in the densities $A_i(y,z)$ are universal in the sense that they are fixed by gauge invariance and kinematics. However they do depend on the choice of the experimental cuts to the photon spectrum and they exhibit a certain ambiguity due to possible different definitions of the function $\Gamma_c(v, m_\ell^2, M^2; M_\gamma)$ introduced in (4.2) (we adopt the form given in (5.1) below). We suggest that these modified densities be used in the data analysis. This does not introduce any model dependence, and takes care of very long distance electromagnetic corrections. Such an experimental analysis of the Dalitz plot density could then provide valuable information on $f_\pm(t,v)$, to be confronted with theoretical calculations of pure QCD and QCD+QED effects.

5 $K_{\ell 3}$ amplitudes at order p^4 and e^2p^2 in CHPT

In this section we analyze the form factors relevant for $K_{\ell 3}$ decays in the framework of CHPT, including electromagnetic corrections at order e^2p^2. As for the non-electromagnetic part, we shall give the results at order p^4 in standard CHPT, including the effect of *strong* isospin breaking ($m_u \neq m_d$). In principle, the order p^6 corrections can be easily included in our formalism, in order to make a more accurate phenomenological analysis.

In Sect. 4 we have introduced the effective form factors $f_\pm(t,v)$, from which the long distance virtual photon effects have been removed [see (4.2)]. We also stressed that in order to give a meaningful definition of such objects one has to specify the function $\Gamma_c(v, m_\ell^2, M^2; M_\gamma)$, arising in perturbation theory from photonic loop contributions to the form factor $F_+(t,v)$:

$$\Gamma_c(v, m_\ell^2, M^2; M_\gamma) = 2M^2 Y \mathcal{C}(v, m_\ell^2, M^2) + 2\log\frac{M m_\ell}{M_\gamma^2}\left(1 + \frac{XY\log X}{\sqrt{R}(1-X^2)}\right). \quad (5.1)$$

In Appendix B we define the auxiliary variables X, R, Y, as well as the function $\mathcal{C}(v, m_\ell^2, M^2)$. Γ_c encodes the leading Coulomb interaction between the charged particles involved in the decay (K^+ and ℓ^+ for $K_{\ell 3}^+$, π^- and ℓ^+ for $K_{\ell 3}^0$). It is IR divergent and singular when the two charged particles are relatively at rest. In the case of K^+ decays, this singularity is outside the physical region, while it occurs on its boundary for the K^0 decay, implying a larger overall effect. It is convenient to write the form factors in a form suitable for a resummation of (potentially large) IR and Coulomb effects. At the same time we want this representation to reproduce the correct CHPT result at order e^2p^2. We achieve this by writing the full form factors as follows, in terms of the effective form factors:

$$F_\pm(t,v) = \left[1 + \frac{\alpha}{4\pi}\Gamma_c(v, m_\ell^2, M^2; M_\gamma)\right] f_\pm(t,v),$$
$$f_\pm(t,v) = \widetilde{f}_\pm(t) + f_\pm^{\mathrm{EM-loc}} + f_\pm^{\mathrm{EM-loop}}(v). \quad (5.2)$$

Let us now comment on each term appearing in (5.2).

(1) $\widetilde{f}_\pm(t)$ represent the pure QCD contributions to the form factors (in principle at any order in the chiral expansion), plus the electromagnetic contributions up to order e^2p^2 generated by the non-derivative Lagrangian defined in (2.1) (proportional to Z). Diagrammatically they arise from purely mesonic graphs. In the definition of $\widetilde{f}_+^{K^+\pi^0}(t)$, we have included also the electromagnetic counterterms relevant to π^0–η mixing [see (5.4) and (5.5) below].
(2) $f_\pm^{\mathrm{EM-loc}}$ represent the local effects of virtual photon exchange.
(3) $f_\pm^{\mathrm{EM-loop}}(v)$ represent the non-local photonic loop contribution (once the Coulomb term has been removed). We have checked that in all relevant cases $f_\pm^{\mathrm{EM-loop}}(v)$ are smooth functions of the variable v in the physical region, allowing one to perform a linear expansion in the Dalitz variables y, z.

Each term in the above decomposition is scale-independent. We now give the full expressions for the form factor components, in terms of loop functions defined in Appendix A.

5.1 The form factors $f_\pm^{K^+\pi^0}(t,u)$

For $f_\pm^{K^+\pi^0}$ the mesonic contribution is given by

$$\widetilde{f}_+(t) = 1 + \sqrt{3}(\varepsilon^{(2)} + \varepsilon_{\mathrm{S}}^{(4)} + \varepsilon_{\mathrm{EM}}^{(4)}) + \frac{1}{2}H_{K^+\pi^0}(t) + \frac{3}{2}H_{K^+\eta}(t) + H_{K^0\pi^-}(t) + \sqrt{3}\varepsilon^{(2)}\left[\frac{5}{2}H_{K\pi}(t) + \frac{1}{2}H_{K\eta}(t)\right]. \quad (5.3)$$

This expression is essentially the pure QCD form factor at $\mathcal{O}(p^4)$ [2], except for the inclusion of electromagnetic contributions to the meson masses and for the additional contribution $\varepsilon_{\mathrm{EM}}^{(4)}$, due to π^0–η mixing at $\mathcal{O}(e^2p^2)$ [13]. The function $H_{PQ}(t)$ [2,4] is reported in Appendix A, and the leading order π^0–η mixing angle $\varepsilon^{(2)}$ has already been given in (2.9). The remaining quantities in (5.3) are given by

$$\varepsilon_{\mathrm{S}}^{(4)} = -\frac{2\varepsilon^{(2)}}{3(4\pi F_0)^2(M_\eta^2 - M_\pi^2)} \times \Big\{(4\pi)^2 64\,[3L_7 + L_8^r(\mu)]\,(M_K^2 - M_\pi^2)^2 - M_\eta^2(M_K^2 - M_\pi^2)\log\frac{M_\eta^2}{\mu^2} + M_\pi^2(M_K^2 - 3M_\pi^2)\log\frac{M_\pi^2}{\mu^2} - 2M_K^2(M_K^2 - 2M_\pi^2)\log\frac{M_K^2}{\mu^2} - 2M_K^2(M_K^2 - M_\pi^2)\Big\}, \quad (5.4)$$

and[3]

$$\varepsilon_{\mathrm{EM}}^{(4)} = \frac{2\sqrt{3}\alpha M_K^2}{108\pi(M_\eta^2 - M_\pi^2)} \times \Big\{2(4\pi)^2[-6K_3^r(\mu) + 3K_4^r(\mu) + 2K_5^r(\mu) + 2K_6^r(\mu)] - 9Z\left(\log\frac{M_K^2}{\mu^2} + 1\right)\Big\}. \quad (5.5)$$

They are related to the one-loop off-diagonal element of the squared mass matrix in the π^0–η sector [13] by

$$\varepsilon_{\mathrm{S}}^{(4)} + \varepsilon_{\mathrm{EM}}^{(4)} = -\frac{M_{\hat\pi^0\hat\eta}^2}{(M_\eta^2 - M_\pi^2)}. \quad (5.6)$$

The analogous contribution for the f_- form factor is given by

$$\widetilde{f}_-(t) = \frac{4}{F_0^2}\left(1 + \frac{\varepsilon^{(2)}}{\sqrt{3}}\right)(M_K^2 - M_\pi^2) \times \left[L_5^r(\mu) - \frac{3}{256\pi^2}\ln\frac{M_{K^\pm}^2}{\mu^2}\right] - \frac{1}{128\pi^2F_0^2}\Big[(3 + \sqrt{3}\varepsilon^{(2)})M_\eta^2\ln\frac{M_\eta^2}{M_{K^\pm}^2} + 2(3 - \sqrt{3}\varepsilon^{(2)})M_{K^0}^2\ln\frac{M_{K^0}^2}{M_{K^\pm}^2} - 2(3 - \sqrt{3}\varepsilon^{(2)})M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{M_{K^\pm}^2} + (1 + 3\sqrt{3}\varepsilon^{(2)})M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{M_{K^\pm}^2}\Big]$$

[3] The last -1 in (5.21) of [13] should be replaced by $+1$; the preceding formula (5.20) is correct. Note also that we neglect terms of order $e^2M_\pi^2$

$$+ \sum_{PQ} \left\{ \left[a_{PQ}(t) + \frac{\Delta_{PQ}}{2t} b_{PQ}(t) \right] K_{PQ}(t) + b_{PQ}(t) \frac{F_0^2}{t} H_{PQ} \right\}. \quad (5.7)$$

The sum in the last line runs over meson pairs occurring in loop diagrams. The functions $K_{PQ}(t)$ and $H_{PQ}(t)$ [4] are reported in Appendix A, while the relevant coefficients $a_{PQ}(t)$ and $b_{PQ}(t)$ are listed in Appendix C.

The local electromagnetic contributions are given by:

$$f_+^{\mathrm{EM-loc}} = 4\pi\alpha \left[2K_{12}^r(\mu) - \frac{8}{3} X_1 - \frac{1}{2} \widetilde{X}_6^r(\mu) - \frac{1}{32\pi^2} \left(3 + \ln \frac{m_\ell^2}{M_{K^\pm}^2} + 3 \ln \frac{M_{K^\pm}^2}{\mu^2} \right) \right], \quad (5.8)$$

$$f_-^{\mathrm{EM-loc}} = 8\pi\alpha \left[2K_3^r(\mu) - K_4^r(\mu) - \frac{1}{3} \left(K_5^r(\mu) + K_6^r(\mu) \right) + X_1 - X_2^r(\mu) + X_3^r(\mu) - \frac{1}{32\pi^2} \left(1 - 2 \ln \frac{m_\ell^2}{M_{K^\pm}^2} - \left(3 + \frac{5Z}{2} \right) \ln \frac{M_{K^\pm}^2}{\mu^2} \right) \right]. \quad (5.9)$$

Finally, the loop contributions are given by

$$f_\pm^{\mathrm{EM-loop}}(u) = \frac{\alpha}{4\pi} [\Gamma_1(u, m_\ell^2, M_K^2) \pm \Gamma_2(u, m_\ell^2, M_K^2)], \quad (5.10)$$

with $\Gamma_{1,2}(v, m_\ell^2, M^2)$ given in Appendix B.

5.2 The form factors $f_\pm^{K^0\pi^-}(t,s)$

In the case of $f_\pm^{K^0\pi^+}$, the mesonic loop contribution is given by

$$\widetilde{f}_+(t) = 1 + \frac{1}{2} H_{K^+\pi^0}(t) + \frac{3}{2} H_{K^+\eta}(t) + H_{K^0\pi^-}(t) + \sqrt{3}\varepsilon^{(2)} \left[H_{K\pi}(t) - H_{K\eta}(t) \right], \quad (5.11)$$

and

$$\widetilde{f}_-(t) = \frac{4}{F_0^2} \left(1 + \frac{2\varepsilon^{(2)}}{\sqrt{3}} \right) (M_K^2 - M_\pi^2) \times \left[L_5^r(\mu) - \frac{3}{256\pi^2} \ln \frac{M_{\pi^\pm}^2}{\mu^2} \right] - \frac{1}{128\pi^2 F_0^2} \left[(3 + 2\sqrt{3}\varepsilon^{(2)}) M_\eta^2 \ln \frac{M_\eta^2}{M_{\pi^\pm}^2} + 2M_{K^0}^2 \ln \frac{M_{K^0}^2}{M_{\pi^\pm}^2} - (3 + 2\sqrt{3}\varepsilon^{(2)}) M_{\pi^0}^2 \ln \frac{M_{\pi^0}^2}{M_{\pi^\pm}^2} \right] + \sum_{PQ} \left\{ \left[c_{PQ}(t) + \frac{\Delta_{PQ}}{2t} d_{PQ}(t) \right] K_{PQ}(t) + d_{PQ}(t) \frac{F_0^2}{t} H_{PQ} \right\}. \quad (5.12)$$

As previously, the sum runs over meson pairs occurring in loop diagrams. The coefficients $c_{PQ}(t)$ and $d_{PQ}(t)$, for the relevant meson pairs, are reported in Appendix C.

The local electromagnetic terms are given by

$$f_+^{\mathrm{EM-loc}} = 4\pi\alpha \left[2K_{12}^r(\mu) + \frac{4}{3} X_1 - \frac{1}{2} \widetilde{X}_6^r(\mu) - \frac{1}{32\pi^2} \left(3 + \ln \frac{m_\ell^2}{M_{\pi^\pm}^2} + 3 \ln \frac{M_{\pi^\pm}^2}{\mu^2} \right) \right], \quad (5.13)$$

$$f_-^{\mathrm{EM-loc}} = 8\pi\alpha \left[-\frac{1}{3} \left(K_5^r(\mu) + K_6^r(\mu) \right) + X_1 + X_2^r(\mu) - X_3^r(\mu) + \frac{1}{32\pi^2} \left(1 - 2 \ln \frac{m_\ell^2}{M_{\pi^\pm}^2} - \left(3 - \frac{Z}{2} \right) \ln \frac{M_{\pi^\pm}^2}{\mu^2} \right) \right]. \quad (5.14)$$

Finally, the loop contributions are

$$f_\pm^{\mathrm{EM-loop}}(s) = \frac{\alpha}{4\pi} [\Gamma_2(s, m_\ell^2, M_\pi^2) \pm \Gamma_1(s, m_\ell^2, M_\pi^2)]. \quad (5.15)$$

5.3 The effective form factors for K_{e3} decays

In the analysis of K_{e3} decays (where only F_+ is observable), in order to avoid unnecessary complications we can write

$$F_+(t, v) = f_+(t) \left\{ 1 + \frac{\alpha}{4\pi} \Gamma_c(v, m_\ell^2, M^2; M_\gamma) + f_+^{\mathrm{EM-loop}}(v) \right\}, \quad (5.16)$$

with

$$f_+(t) = \widetilde{f}_+(t) + f_+^{\mathrm{EM-loc}}. \quad (5.17)$$

Exploiting the ambiguity in the definition of the function Γ_c, here we choose to shift a smooth function of v from the form factor to Γ_c itself. The advantage is that the effective form factor becomes a function of the single variable t. This trick is not possible in the treatment of $K_{\mu 3}$ decays, as the loop contributions $f_+(v)$ and $f_-(v)$ are different.

6 Corrections to kinematical densities

As an illustration of the general considerations of Sect. 4, we present here in detail a possible treatment of the contribution of the real photon emission process

$$K^+(p_K) \to \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu)\gamma(p_\gamma). \quad (6.1)$$

Following the procedure proposed by Ginsberg [9], we define the kinematical variable

$$x = (p_\nu + p_\gamma)^2 = (p_K - p_\pi - p_\ell)^2. \quad (6.2)$$

A possible choice of the purely radiative part of the decay distribution (corresponding to a specific analysis of the experimental data) is to accept all pion and charged lepton energies within the whole K_{e3} Dalitz plot $\mathcal{D}$ and all kinematically allowed values of the Lorentz invariant x defined in (6.2). (The variable x determines the angle between the pion and lepton momentum for given energies E_π, E_ℓ.) This translates into

$$\rho_\gamma(y,z) = \frac{M_K}{2^{12}\pi^5} \int_{M_\gamma^2}^{x_{\max}} \mathrm{d}x \frac{1}{2\pi} \int \frac{\mathrm{d}^3 p_\nu}{p_\nu^0} \frac{\mathrm{d}^3 p_\gamma}{p_\gamma^0} \times \delta^{(4)}(p_K - p_\pi - p_\ell - p_\nu - p_\gamma) \sum_{\mathrm{pol}} |\mathcal{M}^\gamma|^2, \tag{6.3}$$

with

$$x_{\max} = M_K^2 \Big\{ 1 + r_\pi + r_\ell - y - z + \frac{1}{2}\big[yz + \sqrt{(y^2 - 4r_\ell)(z^2 - 4r_\pi)}\big] \Big\}. \tag{6.4}$$

In (6.3) we have extended the integration over the whole range of the invariant mass of the unobserved $\nu_\ell\gamma$ system. The integrals occurring in (6.3) have the general form [9]

$$I_{m,n}(p_1, p_2; P, M_\gamma) := \frac{1}{2\pi} \int \frac{\mathrm{d}^3 q}{q^0} \frac{\mathrm{d}^3 k}{k^0} \frac{\delta^{(4)}(P - q - k)}{(p_1 \cdot k + M_\gamma^2/2)^m (p_2 \cdot k + M_\gamma^2/2)^n}. \tag{6.5}$$

The results for these integrals in the limit $M_\gamma = 0$ can be found in the appendix of [9]. (We have checked these expressions.) Using the definition (6.5), the radiative decay distribution (6.3) can be written as [9]

$$\rho_\gamma(y,z) = \frac{\alpha}{\pi} \Bigg[\rho^{(0)}(y,z) I_0(y,z;M_\gamma) + \frac{G_{\mathrm{F}}^2 |V_{us}|^2 |f_+|^2 M_K}{128\pi^3} \times \int_0^{x_{\max}} \mathrm{d}x \sum_{m,n} c_{m,n} I_{m,n}(p_\ell, p_K; p_K - p_\pi - p_\ell, 0) \Bigg], \tag{6.6}$$

where the infrared divergences are now confined to

$$I_0(y,z;M_\gamma) = \int_{M_\gamma^2}^{x_{\max}} \mathrm{d}x \Big[-2 p_K \cdot p_\ell I_{1,1}(p_\ell, -p_K; p_K - p_\pi - p_\ell; M_\gamma) - M_K^2 I_{0,2}(p_\ell, -p_K; p_K - p_\pi - p_\ell; M_\gamma) - m_\ell^2 I_{2,0}(p_\ell, -p_K; p_K - p_\pi - p_\ell; M_\gamma) \Big]. \tag{6.7}$$

The explicit form of the function I_0 can be found in (27) of [9]. The coefficients $c_{m,n}$ were given in (19) of [9]. Note however the misprint for the values of $c_{-1,0}$ and $c_{1,-2}$ (see the Erratum of [9]). (We have also checked this list of coefficients.)

The function Δ^{IR} introduced in (4.7) can now be related to I_0 by

$$\Delta^{\mathrm{IR}}(y,z) = \frac{\alpha}{\pi} \left[I_0(y,z;M_\gamma) + \frac{1}{2} \Gamma_\ell(u, m_\ell^2, M_K^2; M_\gamma) \right], \tag{6.8}$$

where

$$\Gamma_\ell = \begin{cases} \Gamma_c + \dfrac{4\pi}{\alpha} f_+^{\mathrm{EM-loop}} & \text{for } \ell = e, \\ \Gamma_c & \text{for } \ell = \mu. \end{cases} \tag{6.9}$$

An analytic expression of the integral occurring in (6.6) was given in Appendix B of [11] in terms of the quantities U_i:

$$\int_0^{x_{\max}} \mathrm{d}x \sum_{m,n} c_{m,n} I_{m,n} = \sum_{i=0}^{7} U_i. \tag{6.10}$$

Note that the quantity $J_9(i)$ given in (A9) of [11] (which is needed for the evaluation of U_7) contains two crucial mistakes: the plus-sign in the last line of (A9) should be replaced by a minus-sign, and $|\beta_i^{\max}|$ at the end of the first line of (A9) should simply read $\beta_i^{\max}$. The second error is irrelevant for $\ell = e$ but has disastrous consequences in a certain part of the Dalitz plot for $\ell = \mu$. To the best of our knowledge none of these errors has been reported in an Erratum of [11].

The functions Δ_i^{IB} introduced in (4.7) can now be written as

$$\Delta_1^{\mathrm{IB}} = \frac{2\alpha}{\pi M_K^4} \sum_{i=0}^{7} U_i \Big|_{\xi=0}, \quad \Delta_2^{\mathrm{IB}} = \frac{2\alpha}{\pi M_K^4} \sum_{i=0}^{7} U_i \Big|_{\xi}, \quad \Delta_3^{\mathrm{IB}} = \frac{2\alpha}{\pi M_K^4} \sum_{i=0}^{7} U_i \Big|_{\xi^2}, \tag{6.11}$$

where the symbol ξ used in [11] stands for the ratio f_-/f_+.

7 Application to K_{e3}^+ decay

In this section we provide an illustrative analysis of the K_{e3}^+ decay with inclusion of isospin breaking and radiative corrections. The aim here is to illustrate how one might proceed in order to extract $|V_{us}|$ rather than to give final numbers, which would anyhow still depend on the size of the hitherto unknown two-loop strong interaction contributions. Once they become known, these corrections could then easily be incorporated into the analysis presented here.

As usual we provide a linear expansion for the effective form factor $f_+^{K^+\pi^0}(t)$, which reads

$$f_+^{K^+\pi^0}(t) = f_+^{K^+\pi^0}(0) \left[1 + \frac{t}{M_{\pi^\pm}^2} \widetilde{\lambda}_+ \right], \tag{7.1}$$

with

$$f_+^{K^+\pi^0}(0) = \widetilde{f}_+(0) + f_+^{\mathrm{EM-loc}}, \tag{7.2}$$

and

$$\frac{\widetilde{\lambda}_+}{M_{\pi^\pm}^2} = \frac{\mathrm{d}\widetilde{f}_+(t)}{\mathrm{d}t} \Big|_{t=0}. \tag{7.3}$$

Using the linear expansion for the form factor $f^{K^+\pi^0}(t)$, the decay rate

$$\begin{aligned}&\Gamma(K^+ \to \pi^0 e^+ \nu_e(\gamma)) \\ &:= \Gamma(K^+ \to \pi^0 e^+ \nu_e) + \Gamma(K^+ \to \pi^0 e^+ \nu_e \gamma) \quad (7.4)\end{aligned}$$

can be expressed as

$$\begin{aligned}&\Gamma(K^+ \to \pi^0 e^+ \nu_e(\gamma)) \\ &= \mathcal{N} S_{\rm EW}(M_\rho, M_Z) |f_+^{K^+\pi^0}(0)|^2 I(\widetilde{\lambda}_+), \quad (7.5)\end{aligned}$$

where

$$\begin{aligned}I(\widetilde{\lambda}_+) &= \int_{\mathcal{D}} {\rm d}y{\rm d}z A_1(y,z) \left[1 + \frac{t}{M_{\pi^\pm}^2}\widetilde{\lambda}_+\right]^2 \\ &= a_0 + a_1 \widetilde{\lambda}_+ + a_2 \widetilde{\lambda}_+^2. \quad (7.6)\end{aligned}$$

In (7.5) we have pulled out the short distance enhancement factor $S_{\rm EW}(M_\rho, M_Z)$ from the low-energy constant $X_6^r(\mu)$ [see (2.26)].

In order to extract $|V_{us}|$ we have to provide a theoretical estimate of the form factor at $t = 0$ and the phase-space integral.

7.1 Numerical estimate of $f_+^{K^+\pi^0}(0)$

In our numerical analysis, we use the physical meson masses [1]. We now describe the input used for the other quantities occurring in the form factor expansion.

For the mixing angle defined in (2.9), we are using [20]

$$\varepsilon^{(2)} = (1.061 \pm 0.083) \times 10^{-2}. \quad (7.7)$$

The combination of coupling constants $3L_7 + L_8^r(M_\rho)$ entering in (5.4) can be determined from two observables [4]: the deviation from the Gell-Mann–Okubo mass formula,

$$\Delta_{\rm GMO} = \frac{4M_K^2 - M_\pi^2 - 3M_\eta^2}{M_\eta^2 - M_\pi^2}, \quad (7.8)$$

which is well under control, and a quantity $\Delta_{\rm M}$ related to the $\mathcal{O}(p^4)$ contributions for the ratio

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \widehat{m}}{2\widehat{m}}(1 + \Delta_{\rm M}). \quad (7.9)$$

Combining (10.10) and (10.11) of [4], one finds

$$\begin{aligned}&3L_7 + L_8^r(M_\rho) \\ &= F_0^2 \left\{ -\frac{\Delta_{\rm GMO}}{24(M_\eta^2 - M_\pi^2)} + \frac{\mu_\eta - \mu_\pi - \Delta_{\rm M}}{32(M_K^2 - M_\pi^2)} \right. \\ &\quad \left. + \frac{3M_\eta^2\mu_\eta + M_\pi^2\mu_\pi - 4M_K^2\mu_K}{12(M_\eta^2 - M_\pi^2)^2} \right\}, \quad (7.10)\end{aligned}$$

where

$$\mu_P = \frac{M_P^2}{(4\pi F_0)^2} \log \frac{M_P}{M_\rho}. \quad (7.11)$$

Using

$$\Delta_{\rm M} = 0.065 \pm 0.065 \quad (7.12)$$

from Leutwyler's analysis [20], (7.10) implies

$$3L_7 + L_8^r(M_\rho) = (-0.33 \pm 0.08) \times 10^{-3}, \quad (7.13)$$

with a relatively small error. (Allowing also for higher order corrections one might prefer [21] a more generous bound of, say, $|\Delta_{\rm M}| \le 0.2$ corresponding to $3L_7 + L_8^r(M_\rho) = (-0.25 \pm 0.25) \times 10^{-3}$.)

The relevant combination of electromagnetic coupling constants appearing in (5.5) has been estimated by Bijnens and Prades [22]. For our numerical analysis we add here an error of the typical size $1/(4\pi)^2 \simeq 6.3 \times 10^{-3}$:

$$\begin{aligned}\widehat{K}^r(M_\rho) &:= (-6K_3 + 3K_4 + 2K_5 + 2K_6)^r(M_\rho) \\ &= (5.7 \pm 6.3) \times 10^{-3}. \quad (7.14)\end{aligned}$$

In this way we find the following mesonic contributions to the form factor:

$$\begin{aligned}\widetilde{f}_+(0) &= 1.0002 \left(1 + 0.0228 \cdot \frac{\varepsilon^{(2)} - 1.06 \times 10^{-2}}{1.06 \times 10^{-2}} \right. \\ &\quad + 0.0055 \cdot \frac{3L_7 + L_8^r(M_\rho) + 0.33 \times 10^{-3}}{-0.33 \times 10^{-3}} \\ &\quad \left. + 0.0002 \cdot \frac{\widehat{K}^r(M_\rho) - 5.7 \times 10^{-3}}{5.7 \times 10^{-3}} \right) \\ &= 1.0002 \pm 0.0018 \pm 0.0013 \pm 0.0002 \\ &= 1.0002 \pm 0.0022. \quad (7.15)\end{aligned}$$

In the last line the three individual errors have been added in quadrature. The main contribution to the final error comes from the uncertainties in (7.7) and (7.13). The electromagnetic contribution to π^0–η mixing, entering through (5.5), has very little numerical impact on the central value and the uncertainty.

For the coupling constant K_{12} entering in the purely electromagnetic part (5.8) we use a value extracted from the work of Moussallam [23]:

$$K_{12}^r(M_\rho) = (-4.0 \pm 0.5) \times 10^{-3}. \quad (7.16)$$

For the (unknown) "leptonic" constants we resort to the usual bounds suggested by dimensional analysis:

$$|X_1|,\ |\widetilde{X}_6^r(M_\rho)| \le 6.3 \times 10^{-3}. \quad (7.17)$$

Eventually we find

$$\begin{aligned}f_+^{\rm EM-loc} &= 0.0032 - 0.0007 \cdot \frac{K_{12}^r(M_\rho) + 4 \times 10^{-3}}{-4 \times 10^{-3}} \\ &\quad - 0.0015 \cdot \frac{X_1}{6.3 \times 10^{-3}} - 0.0003 \cdot \frac{\widetilde{X}_6^r(M_\rho)}{6.3 \times 10^{-3}} \\ &= 0.0032 \pm 0.0016. \quad (7.18)\end{aligned}$$

In this contribution, the sizeable relative uncertainty is almost exclusively due to the poor present knowledge of X_1. Despite this, in the final result for $f_+^{K^+\pi^0}(0)$ this is an effect of 0.16%.

Combining the results given above we obtain the final value:

$$f_+^{K^+\pi^0}(0) = 1.0034 \pm 0.0027. \quad (7.19)$$

Table 1. Coefficients entering the phase-space integral

	a_0	a_1	a_2
$\alpha = 0$	0.09653	0.3337	0.4618
$\alpha \neq 0$	0.09533	0.3287	0.4535

7.2 The phase-space factor

The theoretical prediction for the slope parameter is determined by the size of the low-energy constant L_9^r. With

$$L_9^r(M_\rho) = (6.9 \pm 0.7) \times 10^{-3}, \tag{7.20}$$

we find

$$\begin{aligned}\widetilde{\lambda}_+ &= 0.0328 + 0.0321 \cdot \frac{L_9^r(M_\rho) - 6.9 \times 10^{-3}}{6.9 \times 10^{-3}} \\ &= 0.0328 \pm 0.0033. \end{aligned} \tag{7.21}$$

Here one can note the relatively large uncertainty induced by $L_9^r(M_\rho)$. This suggests to use the measured value for the slope when evaluating the phase-space integral.

The numerical coefficients $a_{0,1,2}$ entering in the phase-space expression (7.6) are reported in Table 1. The numbers given here correspond to the specific prescription for the treatment of real photons described in the previous section: accept all pion and charged lepton energies within the whole K_{e3} Dalitz plot $\mathcal{D}$ and all kinematically allowed values of the Lorentz invariant x defined in (6.2). (The variable x determines the angle between the pion and lepton momentum for given energies E_π, E_ℓ.)

The inclusion of radiative corrections in this channel tends to give a negative shift to the phase-space term. Assuming $\widetilde{\lambda}_+ = 0.030$, and evaluating (7.6) with $a_{0,1,2}$ corresponding to $\alpha \neq 0$ and $\alpha = 0$ (see Table 1), one can check that radiative corrections induce in $I(\widetilde{\lambda}_+)$ a negative shift of 1.27%.

7.3 Illustrative extraction of $|V_{us}|$

The Kobayashi–Maskawa matrix element $|V_{us}|$ can be extracted from the K_{e3} decay parameters by

$$|V_{us}| = \frac{16\pi^{3/2}\Gamma(K^+ \to \pi^0 e^+ \nu_e(\gamma))^{1/2}}{G_F M_{K^\pm}^{5/2} S_{EW}(M_\rho, M_Z)^{1/2} |f_+^{K^+\pi^0}(0)| I(\widetilde{\lambda}_+)^{1/2}}. \tag{7.22}$$

The application of this formula requires, of course, experimental numbers for $\Gamma(K^+ \to \pi^0 e^+ \nu_e(\gamma))$ and $\widetilde{\lambda}_+$ where the emission of real photons has been taken into account in accordance with the prescription given above. As such a consistent treatment of the radiative corrections cannot be guaranteed for all the experimental data included in the present values [1] of the K_{e3} decay parameters given by the Particle Data Group (PDG), the following analysis should only be regarded as a preliminary illustration of the numerics. An up-to-date determination of $|V_{us}|$ with the highest possible precision will then be the task of future high statistics K_{e3} measurements, and will require the consistent inclusion of $\mathcal{O}(p^6)$ terms in the form factor.

Using [1]

$$\begin{aligned}\Gamma(K^+ \to \pi^0 e^+ \nu_e(\gamma)) &= (2.56 \pm 0.03) \times 10^{-15}\text{MeV}, \\ \widetilde{\lambda}_+ &= 0.0276 \pm 0.0021\end{aligned} \tag{7.23}$$

for the K_{e3} parameters, $S_{EW}(M_\rho, M_Z) = 1.0232$ for the short distance enhancement factor [7] (leading logarithmic and QCD corrections included), and the $\mathcal{O}(p^4)$ prediction (7.19) for $f_+^{K^+\pi^0}(0)$, we find

$$\begin{aligned}|V_{us}| &= 0.2173 \pm 0.0013 \pm 0.0008 \pm 0.0006 \\ &= 0.2173 \pm 0.0016,\end{aligned} \tag{7.24}$$

where the errors correspond to

$$\begin{aligned}\Delta|V_{us}| &= |V_{us}| \left(\pm \frac{1}{2}\frac{\Delta\Gamma}{\Gamma} \pm 0.05 \cdot \frac{\Delta\widetilde{\lambda}_+}{\widetilde{\lambda}_+} \pm \frac{\Delta f_+(0)}{f_+(0)} \right) \\ &= |V_{us}| \big(\pm 0.6\% \pm 0.4\% \pm 0.3\% \big).\end{aligned} \tag{7.25}$$

So far, our numerical estimates were based on the theoretical result for the form factor up to $\mathcal{O}(p^4)$ in the strong part and to $\mathcal{O}(e^2p^2)$ for the electromagnetic contributions. In the case of the pion scattering lengths, it has been shown explicitly [24] that the strong interaction contributions of $\mathcal{O}(p^6)$ are of comparable size to the electromagnetic corrections. (See also [25] for a discussion of electromagnetic effects in neutral pion scattering.) A similar feature can also be expected for the $K_{\ell 3}$ decays. Although a calculation of isospin conserving corrections of $\mathcal{O}(p^6)$ is under way [16], for the time being we have to be content with the rough estimate [5]

$$f_+^{K^+\pi^0}\Big|_{p^6} = -0.016 \pm 0.008, \tag{7.26}$$

given by Leutwyler and Roos already some years ago. Adding[4] (7.19) and (7.26), we obtain

$$f_+^{K^+\pi^0}(0) = 0.9874 \pm 0.0084, \tag{7.27}$$

and, consequently,

$$\begin{aligned}|V_{us}| &= 0.2207 \pm 0.0013 \pm 0.0008 \pm 0.0019 \\ &= 0.2207 \pm 0.0024,\end{aligned} \tag{7.28}$$

which is in agreement with the current PDG value [1],

$$|V_{us}| = 0.2196 \pm 0.0023, \tag{7.29}$$

taken from the analysis in [5].

On the other hand, the unitarity of the Kobayashi–Maskawa mixing matrix together with the present experimental value [1] for $|V_{ud}|$ implies

$$|V_{us}| = 0.2287 \pm 0.0034, \tag{7.30}$$

[4] Also the input parameters in the strong $\mathcal{O}(p^4)$ part of the form factor will receive appropriate shifts [26] in a complete and consistent analysis of $\mathcal{O}(p^6)$ effects

indicating a possible problem for three-generation mixing. Whether this discrepancy will survive once two-loop chiral corrections are included remains to be seen. It is interesting to notice that although these corrections would have to be about twice the size of the estimate (7.26) in order to maintain the unitarity of the CKM matrix, they still would not be unnaturally large from the point of view of the chiral expansion, and could be accounted for by $\mathcal{O}(m_s^2)$ contributions [27].

8 Conclusions

In the present work, we have presented general formulae for the $K_{\ell 3}$ form factors at one loop in the presence of radiative corrections. Two features induced by the electromagnetic interactions are worth noticing. First, the form factors now depend not only on the momentum transferred between the kaon and pion, but also on a second kinematical variable. Second, in addition to the usual electromagnetic low-energy constants (K_i's of [12]), there are contributions from four of the local counterterms X_i introduced in [15]. These counterterms are specific to semileptonic processes of pseudoscalar mesons and renormalize the ultraviolet divergences induced by the exchange of virtual photons between charged mesons and leptons.

Loops with virtual photons generate also infrared divergences. In order to deal with them, we have analyzed the associated real photon emission processes. We have given a general description of the changes induced in the Dalitz plot density, and have proposed a model independent procedure for including radiative corrections in the data analysis. This consists in incorporating the known long distance electromagnetic effects into generalized kinematical densities, while including all the structure dependent (UV sensitive) terms as corrections to the form factors. Details of the new kinematical densities will depend eventually on the way the specific experimental set-up deals with real photon emission. For a quite generic set-up configuration (see Sect. 6), we have rederived explicit expressions for the K^+_{e3} mode, correcting some mistakes in earlier work [11].

Within this framework, we have studied the effect of radiative corrections in the extraction of the CKM matrix element $|V_{us}|$ from the K^+_{e3} mode. As compared to the pure $\mathcal{O}(p^4)$ form factors [2], the inclusion of $\mathcal{O}(e^2p^2)$ electromagnetic contributions shifts $f_+(0)$ by about (0.36 ± 0.16)%. Moreover, the radiative corrections produce an effective reduction of -1.27% for the phase-space integral. We note here that the uncertainty on the form factor up to $\mathcal{O}(p^4, e^2p^2)$ is well under control, and it affects the extraction of $|V_{us}|$ only marginally compared to present experimental errors. This feature persists in the case of K^0_{e3} mode (not analyzed numerically in this work), where the hadronic uncertainties are even less effective. We think this is a relevant result of our analysis, opening the road to a precision determination of $|V_{us}|$, for which the next two important ingredients are

(1) from the theoretical side: the inclusion of two-loop chiral corrections, and
(2) from the experimental side: a new high statistics measurement of branching ratios and slope parameters, including radiative corrections in the model independent way outlined in this work.

Acknowledgements. We would like to thank G. Ecker for fruitful discussions and a careful reading of the manuscript. We are grateful to H. Pichl for useful comments.

Appendix

A Mesonic loop functions

The loop function $H_{PQ}(t)$ [2,4] is given by

$$H_{PQ}(t) = \frac{1}{F^2}\left[h^r_{PQ}(t,\mu) + \frac{2}{3}tL^r_9(\mu)\right], \qquad (A.1)$$

where

$$\begin{aligned} h^r_{PQ}(t,\mu) = &\frac{1}{12t}\lambda(t, M_P^2, M_Q^2)\bar{J}_{PQ}(t) \\ &+ \frac{1}{18(4\pi)^2}(t - 3\Sigma_{PQ}) \\ &- \frac{1}{12}\left\{\frac{2\Sigma_{PQ} - t}{\Delta_{PQ}}[A_P(\mu) - A_Q(\mu)] \right. \\ &\left. - 2[A_P(\mu) + A_Q(\mu)]\right\}, \end{aligned} \qquad (A.2)$$

with

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz), \qquad (A.3)$$

$$\Sigma_{PQ} = M_P^2 + M_Q^2, \quad \Delta_{PQ} = M_P^2 - M_Q^2, \qquad (A.4)$$

$$A_P(\mu) = -\frac{M_P^2}{(4\pi)^2}\log\frac{M_P^2}{\mu^2}, \qquad (A.5)$$

and

$$\begin{aligned} \bar{J}_{PQ}(t) = \frac{1}{32\pi^2}&\left[2 + \frac{\Delta_{PQ}}{t}\log\frac{M_Q^2}{M_P^2} - \frac{\Sigma_{PQ}}{\Delta_{PQ}}\log\frac{M_Q^2}{M_P^2} \right. \\ &- \frac{\lambda^{1/2}(t, M_P^2, M_Q^2)}{t} \\ &\left. \times \log\left(\frac{[t + \lambda^{1/2}(t, M_P^2, M_Q^2)]^2 - \Delta_{PQ}^2}{[t - \lambda^{1/2}(t, M_P^2, M_Q^2)]^2 - \Delta_{PQ}^2}\right)\right]. \end{aligned} \qquad (A.6)$$

Moreover, in the expansion of the form factors $\widetilde{f}_-(t)$, one needs the function

$$K_{PQ}(t) = \frac{\Delta_{PQ}}{2t}\bar{J}_{PQ}(t). \qquad (A.7)$$

B Photonic loop functions

The photonic loop contributions to the $K_{\ell 3}$ form factors depend on the charged lepton and meson masses m_ℓ^2, M^2, as well as on the Mandelstam variables $u=(p_K-p_\ell)^2$ (for $K^+_{\ell 3}$ decays) and $s=(p_\pi+p_\ell)^2$ (for $K^0_{\ell 3}$ decays). In what follows we denote by v the Mandelstam variable appropriate to each decay. In order to express the loop functions in a compact way, it is useful to define the following intermediate variables:

$$R=\frac{m_\ell^2}{M^2},\quad Y=1+R-\frac{v}{M^2},\quad X=\frac{Y-\sqrt{Y^2-4R}}{2\sqrt{R}}. \tag{B.1}$$

In terms of such variables and of the dilogarithm

$$\mathrm{Li}_2(x)=-\int_0^1\frac{\mathrm{d}t}{t}\log(1-xt), \tag{B.2}$$

the functions contributing to $\Gamma_c(v,m_\ell^2,M^2;M_\gamma)$ and $f_\pm^{\mathrm{EM-loop}}(v)$ are given by

$$\begin{aligned}\mathcal{C}(v,m_\ell^2,M^2)&=\frac{1}{m_\ell M}\frac{X}{1-X^2} \\ &\times\left[-\frac{1}{2}\log^2 X+2\log X\log(1-X^2)-\frac{\pi^2}{6}+\frac{1}{8}\log^2 R\right.\\ &\left.+\mathrm{Li}_2(X^2)+\mathrm{Li}_2\left(1-\frac{X}{\sqrt{R}}\right)+\mathrm{Li}_2(1-X\sqrt{R})\right],\end{aligned} \tag{B.3}$$

$$\begin{aligned}\Gamma_1(v,m_\ell^2,M^2)&=\frac{1}{2}[-\ln R+(4-3Y)\mathcal{F}(v,m_\ell^2,M^2)],\\ \Gamma_2(v,m_\ell^2,M^2)&=\frac{1}{2}\left(1-\frac{m_\ell^2}{u}\right)\\ &\quad\times[-\mathcal{F}(v,m_\ell^2,M^2)(1-R)+\ln R]\\ &\quad-\frac{1}{2}(3-Y)\mathcal{F}(v,m_\ell^2,M^2),\end{aligned} \tag{B.4}$$

and

$$\mathcal{F}(v,m_\ell^2,M^2)=\frac{2}{\sqrt{R}}\frac{X}{1-X^2}\ln X. \tag{B.5}$$

C Coefficients entering $\widetilde{f}_-^{K^+\pi^0}$ and $\widetilde{f}_-^{K^0\pi^-}$

In this section we report the coefficients $a_{PQ}(t)$, $b_{PQ}(t)$, $c_{PQ}(t)$, and $d_{PQ}(t)$, appearing in the expression of $\widetilde{f}_-^{K^+\pi^0}$ and $\widetilde{f}_-^{K^0\pi^-}$ ((5.7) and (5.12)):

$$\begin{aligned}a_{K^+\pi^0}(t)&=\frac{2M_K^2+2M_\pi^2-t}{4F_0^2}\\ &\quad+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2+22M_\pi^2-9t}{4F_0^2}+4\pi\alpha Z,\\ a_{K^0\pi^-}(t)&=\frac{-2M_K^2-2M_\pi^2+3t}{2F_0^2}\\ &\quad+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2+6M_\pi^2-3t}{2F_0^2}-16\pi\alpha Z,\\ a_{K^+\eta}(t)&=\frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}\\ &\quad+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{6M_K^2-2M_\pi^2-3t}{4F_0^2}+12\pi\alpha Z.\end{aligned} \tag{C.1}$$

$$\begin{aligned}b_{K^+\pi^0}(t)&=-\frac{M_K^2-M_\pi^2}{2F_0^2}-\left(\frac{7\varepsilon^{(2)}}{2\sqrt{3}}\right)\frac{M_K^2-M_\pi^2}{F_0^2}-4\pi\alpha Z,\\ b_{K^0\pi^-}(t)&=-\frac{M_K^2-M_\pi^2}{F_0^2}-\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{M_K^2-M_\pi^2}{F_0^2}-8\pi\alpha Z,\\ b_{K^+\eta}(t)&=-\frac{3}{2}\frac{M_K^2-M_\pi^2}{F_0^2}+\left(\frac{\sqrt{3}\varepsilon^{(2)}}{2}\right)\frac{M_K^2-M_\pi^2}{F_0^2}\\ &\quad-12\pi\alpha Z.\end{aligned} \tag{C.2}$$

$$\begin{aligned}c_{K^+\pi^0}(t)&=-\frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-4M_K^2+3t}{2F_0^2}\\ &\quad-8\pi\alpha Z,\\ c_{K^0\pi^-}(t)&=\frac{t}{2F_0^2},\\ c_{K^+\eta}(t)&=\frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{4M_K^2-3t}{2F_0^2}.\end{aligned} \tag{C.3}$$

$$\begin{aligned}d_{K^+\pi^0}(t)&=-\frac{M_K^2-M_\pi^2}{2F_0^2}-\left(\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{M_K^2-M_\pi^2}{F_0^2}+4\pi\alpha Z,\\ d_{K^0\pi^-}(t)&=-\frac{M_K^2-M_\pi^2}{F_0^2}-\left(\frac{2\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{M_K^2-M_\pi^2}{F_0^2}+8\pi\alpha Z,\\ d_{K^+\eta}(t)&=-\frac{3}{2}\frac{M_K^2-M_\pi^2}{F_0^2}+12\pi\alpha Z.\end{aligned} \tag{C.4}$$

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